

Entanglement evolution in multipartite cavity-reservoir systems under local unitary operations

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We analyze the entanglement evolution of two cavity photons being affected by the dissipation of two individual reservoirs. Under an arbitrary local unitary operation on the initial state, it is shown that there is only one parameter which changes the entanglement dynamics. For the bipartite subsystems, we show that the entanglement of the cavity photons is correlated with that of the reservoirs, although the local operation can delay the time at which the photon entanglement disappears and advance the time at which the reservoir entanglement appears. Furthermore, via a new defined four-qubit entanglement measure and two three-qubit entanglement measures, we study the multipartite entanglement evolution in the composite system, which allows us to analyze quantitatively both bipartite and multipartite entanglement within a unified framework. In addition, we also discuss the entanglement evolution with an arbitrary initial state.

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I. INTRODUCTION

As one of the most subtle phenomena in many-body systems, quantum entanglement has now been an important physical resource widely used in quantum communication and quantum computation [1, 2]. Therefore it is fundamental to characterize the entanglement nature in quantum systems, especially at a quantitative level. Till now, although bipartite entanglement is well understood in many aspects, the entanglement in multipartite systems is far from clear and thus deserve further exploration.

Entanglement dynamical behavior is an important property in practical quantum information processing. This is because entanglement is fragile and always decays due to unwanted interactions between the system and its environment. A theoretical study of two-atom spontaneous emission shows that entanglement does not always decay in an asymptotic way and it can be corrupted in a finite time [3], which is referred to as entanglement sudden death (ESD). Some earlier studies also pointed out this fact that even a very weakly dissipative environment can disentangle the quantum system in a finite time [4–7]. The ESD phenomenon has recently received a lot of attentions [8–15] (see also a review paper [16] and references therein), and, experimentally, it has been detected in photon [17] and atom systems [18].

A deep understanding on the ESD phenomenon concerns the problem where the lost entanglement goes. To answer the question, it is proper to enlarge the system to include its environment. Recently, López *et al* analyzed the entanglement evolution in a composite system consisting of entangled cavity photons with individual reservoirs [19], and show that the entanglement sudden birth (ESB) of reservoir-reservoir subsystem must happen whenever the ESD of cavity-cavity sub-

system occurs. Moreover, in Ref. [20], Bai *et al* presented a entanglement monogamy relation in multipartite systems and analyzed quantitatively the bipartite entanglement transfer in the multipartite cavity-reservoir system.

However, in the above analysis, the multipartite entanglement in the composite cavity-reservoir system is not well characterized, although the residual entanglement [20] can indicate its existence. Moreover, the authors only consider the symmetric initial state like $|\phi\rangle = \alpha|00\rangle + \beta|11\rangle$. When the initial state is asymmetric, the entanglement evolution can be very different. For example, a σ_x operation acting on the symmetric state can change the evolution of the entangled cavity photons from the ESD route to the asymptotic decay route, although the two kinds of initial states have the equal entanglement. Therefore, it is desirable to consider the entanglement dynamical behavior for the asymmetric case and, particularly, find a good entanglement measure to characterize the genuine multipartite entanglement evolution.

In this paper, for the asymmetric initial state modulated by an arbitrary local unitary (LU) operation, we analyze its entanglement evolution in the multipartite cavity-reservoir system. In Sec. II, we derive the effective output state under the LU operation, in which there is only one parameter affecting the entanglement dynamics. In Sec. III, we analyze the bipartite entanglement transfer in the composite system, and point out the cavity photon entanglement is still correlated with the reservoir entanglement although the local operation can delay the ESD time and advance the ESB time. In Sec. IV, the multipartite entanglement evolution is studied via a new defined four-qubit entanglement measure and two three-qubit entanglement measures. In Sec. V, within a unified framework, we investigate the relation between bipartite entanglement transfer and multipartite entanglement transition in the composite system. Finally, we discuss the entanglement evolution with an arbitrary initial state and give a brief conclusion in Sec. VI.

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II. THE EFFECTIVE OUTPUT STATE UNDER THE LU OPERATION

Before the derivation of the effective output state under the LU operation, we first recall the multipartite cavity-reservoir system. In Ref. [19], López *et al* considered two entangled cavity photons being affected by the dissipation of two individual N -mode reservoirs where the interaction of a single cavity-reservoir system is described by the Hamiltonian

$$\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\sum_{k=1}^N\omega_k\hat{b}_k^\dagger\hat{b}_k + \hbar\sum_{k=1}^Ng_k(\hat{a}\hat{b}_k^\dagger + \hat{b}_k\hat{a}^\dagger). \quad (1)$$

The authors analyzed the entanglement evolution with the symmetric initial state

$$|\Phi_0\rangle = (\alpha|00\rangle + \beta|11\rangle)_{c_1c_2}|00\rangle_{r_1r_2}, \quad (2)$$

in which the reservoirs are in the vacuum state and the quantum state of cavity photons is invariant under the permutation of the qubits c_1 and c_2 . They show that, along the time evolution, the ESD of two photons can happen when the initial state amplitudes satisfy the condition $\alpha < \beta$, and this procedure is necessarily related to the ESB of two reservoirs.

Now, we consider the asymmetric initial state modulated by an arbitrary single-qubit LU operation. Without loss of generality, we assume that the operation acts on the first cavity, and then the initial state can be written as

$$|\Phi_0^a\rangle = U_{c_1}|\Phi_0\rangle. \quad (3)$$

For an arbitrary single qubit LU operation, one can decompose it as [21]

$$U(\zeta, \eta, \gamma, \delta) = e^{i\zeta}R_z(\eta)R_y(\gamma)R_z(\delta), \quad (4)$$

where the $e^{i\zeta}$ is a global phase shift and $R_k(\theta) = \exp(-i\theta\sigma_k/2)$ is the rotation along the $k(=y, z)$ axis with the σ_k being the Pauli matrix. In this case, the output state under the time evolution is

$$\begin{aligned} |\Phi_t\rangle &= U_{c_1r_1}(\hat{H}, t) \otimes U_{c_2r_2}(\hat{H}, t)|\Phi_0^a\rangle \\ &\simeq U_{c_1r_1}(\hat{H}', t) \otimes U_{c_2r_2}(\hat{H}'', t)[R_y(\gamma)_{c_1}|\Phi_0\rangle], \end{aligned} \quad (5)$$

where $\hat{H}' = R_z^\dagger(\eta)_{c_1}\hat{H}R_z(\eta)_{c_1}$, $\hat{H}'' = R_z^\dagger(\delta)_{c_2}\hat{H}R_z(\delta)_{c_2}$, and the \simeq means the states on two sides are equivalent up to some LU operations (for a detail derivation, see the appendix). After considering the effect of the evolution $U_{c_1r_1}(\hat{H}', t)$ on the entanglement dynamics, we find that it is equivalent to that of the evolution $U_{c_1r_1}(\hat{H}, t)$ (in the appendix, we give the proof). The case for the evolution $U_{c_2r_2}(\hat{H}'', t)$ is similar. Then Eq. (5) can be rewritten as

$$|\Phi_t\rangle \simeq U_{c_1r_1}(\hat{H}, t) \otimes U_{c_2r_2}(\hat{H}, t)[R_y(\gamma)_{c_1}|\Phi_0\rangle], \quad (6)$$

which means that, under an arbitrary LU operation $U_{c_1}(\zeta, \eta, \gamma, \delta)$, the entanglement evolution is only sensitive to the rotation $R_y(\gamma)_{c_1}$.

Therefore, the effective initial state for the entanglement evolution is

$$|\Psi_0\rangle = R_y(\gamma)_{c_1}|\Phi_0\rangle = (\alpha|\tilde{0}0\rangle + \beta|\tilde{1}1\rangle)_{c_1c_2}|00\rangle_{r_1r_2}, \quad (7)$$

in which the new basic vectors are $|\tilde{0}\rangle = \cos(\gamma/2)|0\rangle + \sin(\gamma/2)|1\rangle$ and $|\tilde{1}\rangle = -\sin(\gamma/2)|0\rangle + \cos(\gamma/2)|1\rangle$. For the output state, we use the approximation [19]

$$U(\hat{H}, t)_{cr}|10\rangle = \xi|10\rangle + \chi|01\rangle, \quad (8)$$

where the amplitudes are $\xi(t) = \exp(-\kappa t/2)$ and $\chi(t) = [1 - \exp(-\kappa t)]^{1/2}$ in the limit of $N \rightarrow \infty$ for a reservoir with a flat spectrum. Then the effective output state has the form

$$\begin{aligned} |\Psi_t\rangle &= \alpha(\cos\frac{\gamma}{2}|00\rangle + \sin\frac{\gamma}{2}|\phi_t\rangle)_{c_1r_1}|00\rangle_{c_2r_2} \\ &\quad - \beta(\sin\frac{\gamma}{2}|00\rangle - \cos\frac{\gamma}{2}|\phi_t\rangle)_{c_1r_1}|\phi_t\rangle_{c_2r_2}, \end{aligned} \quad (9)$$

where $|\phi_t\rangle = \xi(t)|10\rangle_{cr} + \chi(t)|01\rangle_{cr}$ and the parameter γ being chosen in the range $[0, \pi]$.

III. TWO-QUBIT ENTANGLEMENT EVOLUTION UNDER THE LU OPERATION

According to the effective output state $|\Psi_t\rangle$ in Eq. (9), we can derive the density matrices of different subsystems and analyze their entanglement dynamical behaviors. We first consider the subsystem of two cavity photons, for which its density matrix is

$$\rho_{c_1c_2}(t) = \psi_1 + \psi_2 + \psi_3 + \psi_4, \quad (10)$$

where $\psi_i = |\psi_i\rangle\langle\psi_i|$ and the four non-normalized pure state components are $|\psi_1\rangle = \alpha\cos(\gamma/2)|00\rangle + \alpha\sin(\gamma/2)\xi|10\rangle - \beta\sin(\gamma/2)\xi|01\rangle + \beta\cos(\gamma/2)\xi^2|11\rangle$, $|\psi_2\rangle = \beta\sin(\gamma/2)\chi|00\rangle - \beta\cos(\gamma/2)\xi\chi|10\rangle$, $|\psi_3\rangle = \alpha\sin(\gamma/2)\chi|00\rangle + \beta\cos(\gamma/2)\xi\chi|01\rangle$, and $|\psi_4\rangle = \beta\cos(\gamma/2)\chi^2|00\rangle$, respectively. For the two reservoirs, its density matrix is similar to that of the cavity photons and the following relation holds

$$\rho_{r_1r_2}(t) = S_{\xi\leftrightarrow\chi}[\rho_{c_1c_2}(t)], \quad (11)$$

where $S_{\xi\leftrightarrow\chi}$ exchanges the parameters ξ and χ (*i.e.*, $\xi \rightarrow \chi$ and $\chi \rightarrow \xi$).

Based on the previous analysis in Ref. [20], we choose the square of the concurrence to characterize the two-qubit entanglement evolution. The concurrence is defined as [22] $C(\rho_{ij}) = \max(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4})$ with the decreasing nonnegative real numbers λ_i being the eigenvalues of the matrix $R_{ij} = \rho_{ij}(\sigma_y \otimes \sigma_y)\rho_{ij}^*(\sigma_y \otimes \sigma_y)$. After computing the eigenvalues of the matrices $R_{c_1c_2}$ and $R_{r_1r_2}$ [23], we can obtain

$$\begin{aligned} C_{c_1c_2}^2(t) &= 4[\max(|\alpha\beta\xi^2| - |\beta\xi\chi|^2\cos^2(\gamma/2), 0)]^2, \\ C_{r_1r_2}^2(t) &= 4[\max(|\alpha\beta\chi^2| - |\beta\xi\chi|^2\cos^2(\gamma/2), 0)]^2 \end{aligned} \quad (12)$$

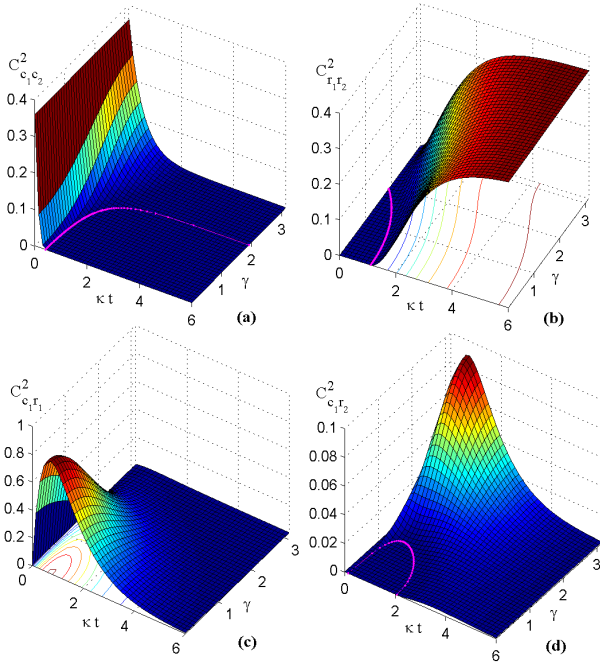


FIG. 1: (Color online) Different two-qubit entanglement as a function of the time evolution κt and the rotation parameter γ .

Combining the concurrences and the expressions of $\xi(t)$ and $\chi(t)$ in Eq. (8), we know that, for a given asymmetric initial state (given the parameters α , β and γ), the cavity photons entanglement decreases and the reservoir entanglement increases along with the time evolution.

It is still an unsolved problem whether or not the ESD of cavity photons and the ESB of the reservoirs are correlated in the asymmetric case. With the conditions $C_{c_1 c_2}(t) = 0$ and $C_{r_1 r_2}(t) = 0$, we can deduce the times of the ESD and the ESB, which have the forms

$$\begin{aligned} t_{ESD}(\rho_{c_1 c_2}) &= -\frac{1}{\kappa} \ln \left(1 - \frac{\alpha}{\beta \cdot \cos^2(\gamma/2)} \right), \\ t_{ESB}(\rho_{r_1 r_2}) &= \frac{1}{\kappa} \ln \frac{\beta \cdot \cos^2(\gamma/2)}{\alpha}, \end{aligned} \quad (13)$$

where the parameter κ is the dissipative constant (note that $\xi = \exp(-\kappa t/2)$ in the entanglement evolution). According to the two times, we can derive that the ESD of two photons occurs when $\beta \cdot \cos^2(\gamma/2) > \alpha$, as is the case for the ESB of two reservoirs. This means that the correlation between the ESD and the ESB *still* holds for the asymmetric initial states, i.e., the ESB of the reservoirs must happen when the ESD of cavity photons occurs.

As an example, we choose the initial state parameters as $\alpha = 1/\sqrt{10}$ and $\beta = 3/\sqrt{10}$. In Fig. 1(a), the concurrence $C_{c_1 c_2}^2$ is plotted as a function of the time κt and the rotation parameter γ . For a fixed value of κt , the photon entanglement increases with the parameter γ . When the γ is given, the $C_{c_1 c_2}^2$ decreases along the time κt . The ESD line (the purple line) is also plotted in the figure, where the γ can delay the ESD time.

It is interesting that the entanglement evolution changes to the asymptotical decay route before the γ attains to the value π , and the critical value is $\gamma = 2\arccos\sqrt{1/3} \approx 1.91063$. In Fig. 1(b), the entanglement evolution of $C_{r_1 r_2}^2$ is plotted, where the parameter γ can increase the reservoir entanglement and advance the ESB time (the purple line). The critical value for the route transition is also $\gamma = 2\arccos\sqrt{1/3}$. Moreover, depending on the value of the γ , the ESB can manifest before, simultaneously and after the ESD.

The two-qubit entanglement of subsystem $c_1 r_1$ has the form

$$C_{c_1 r_1}^2(t) = \xi^2 \chi^2 [1 + (\beta^2 - \alpha^2) \cos(\gamma)]^2. \quad (14)$$

In Fig. 1(c), the concurrence is plotted as a function of the parameters κt and γ . The maximum of $C_{c_1 r_1}^2(t)$ appears at the time $\kappa t = \ln 2$, and the entanglement decreases with the γ . However, the γ does not change the entanglement of subsystem $c_2 r_2$, because the rotation $R_y(\gamma)$ acts on the first cavity. For the subsystems $c_1 r_2$ and $c_2 r_1$, we can get that they have the equal entanglement, which can be expressed as

$$C_{c_1 r_2}^2(t) = 4[\max(|\alpha\beta\xi\chi| - |\beta\xi\chi|^2 \cos^2(\gamma/2), 0)]^2. \quad (15)$$

In Fig. 1(d), the concurrence is plotted. When $\gamma = 0$, the entanglement evolution experiences the ESD at the time $\kappa t = \ln[3(3 - \sqrt{5})/2]$ and the ESB at the time $\kappa t = \ln[3(3 + \sqrt{5})/2]$ (the two intersections between the purple line and the κt axis), then the entanglement changes asymptotically. Along with the increase of the parameter γ , the time window between the ESD and ESB decreases, and the window become a point when $\gamma = 2\arccos\sqrt{2/3} \approx 1.23096$. After this value, both the ESD and the ESB phenomena disappear.

IV. MULTIPARTITE ENTANGLEMENT EVOLUTION UNDER THE LU OPERATION

Before analyzing the entanglement evolution, we first consider how to characterize the multipartite entanglement in the composite system. In Ref. [19], the multipartite concurrence C_N [24] can not characterize completely the genuine multipartite entanglement, due to its nonzero value for two Bell states. In Ref. [20], it is shown that the genuine multipartite entanglement can be indicated by the two-qubit residual entanglement

$$M_{c_1 r_1}(\Phi_t) = C_{c_1 r_1 | c_2 r_2}^2(t) - \sum C_{i' j'}^2(t), \quad (16)$$

where the sum subscripts $i' \in \{c_1, r_1\}$ and $j' \in \{c_2, r_2\}$, respectively. However, the entanglement monotone property of $M_{c_1 r_1}$ is not clear, even for the case of symmetric initial states.

The average multipartite entanglement may quantify the genuine multiqubit entanglement based on much numerical analysis, which is defined as [25]

$$E_{ms}(\Psi_4) = \frac{\sum_i \tau_i(\rho_i) - 2 \sum_{i>j} C_{ij}^2(\rho_{ij})}{4}, \quad (17)$$

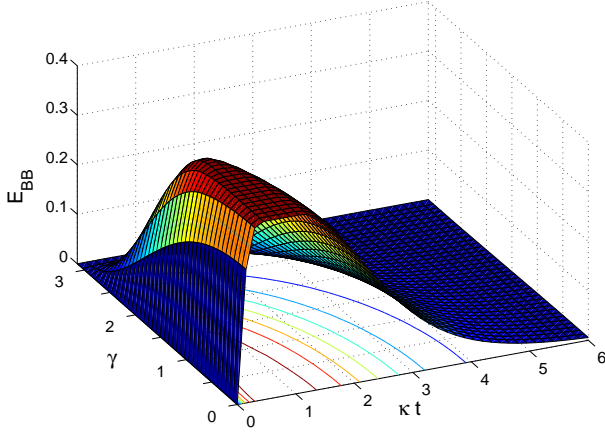


FIG. 2: (Color online) Block-block entanglement as a function of the parameters κt and γ .

where the $\tau_i = 2(1 - \text{tr}\rho_i^2)$ is the linear entropy and the C_{ij} is the concurrence. For the four-qubit cluster-class states, an analytical proof of the entanglement monotone property for the E_{ms} was given in Refs. [26, 27].

For the effective output state $|\Psi_t\rangle$ in Eq. (9), we can compute the average multipartite entanglement E_{ms} and the residual entanglement $M_{c_1r_1}$. After comparing the two measures, we can obtain that they are equivalent up to a constant factor 2. Therefore, we can define entanglement measure

$$E_{BB}(\Psi_t) = M_{c_1r_1}(\Psi_t) = 2E_{ms}(\Psi_t), \quad (18)$$

which quantifies the genuine multipartite entanglement between the blocks c_1r_1 and c_2r_2 and its entanglement monotone property is based on the numerical analysis on the average multipartite entanglement E_{ms} . In Fig. 2, the E_{BB} is plotted as a function of the parameters κt and γ , where the initial state parameters are chosen as $\alpha = 1/\sqrt{10}$ and $\beta = 3/\sqrt{10}$. When $\gamma = 0$, the $E_{BB}(\kappa t)$ increases from 0 to 0.36 in the region $\kappa t \in \{0, \ln(3/2)\}$, then it keeps invariant until the time $\kappa t = \ln 3$, finally, the $E_{BB}(\kappa t)$ decreases asymptotically. With the increase of the γ , the width of the plateau decreases and can be expressed as

$$t_w = \ln[3\cos^2(\gamma/2) - 1]. \quad (19)$$

When $\gamma = 2\arccos(\sqrt{2/3})$, the width changes to zero and the evolution time is $\kappa t = \ln 2$. After this value, the block-block entanglement decreases along with the γ , and vanishes when $\gamma = \pi$.

The genuine tripartite entanglement in the composite system can be quantified by the mixed state three-tangle [28]

$$\tau_3(\rho_{ijk}) = \min_{\{p_x, \varphi_{ijk}^{(x)}\}} \sum p_x \tau(\varphi_{ijk}^{(x)}), \quad (20)$$

where $\tau(\varphi_{ijk}^{(x)}) = \tau_i - C_{ij}^2 - C_{ik}^2$ [29] is the pure state three-tangle and the minimum runs over all the pure state decompositions of ρ_{ijk} . The reduced density matrix of subsystem

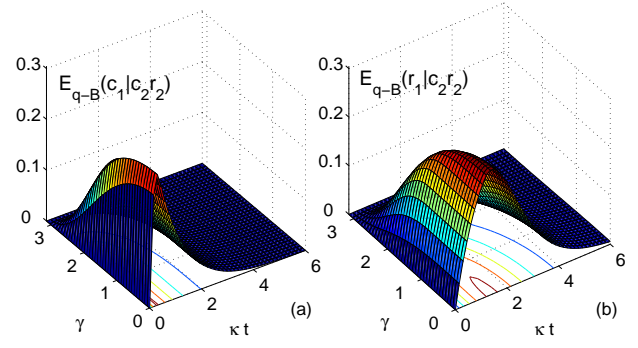


FIG. 3: (Color online) (a) Qubit-block entanglement $E_{q-B}(c_1|c_2r_2)$ and (b) qubit-block entanglement $E_{q-B}(r_1|c_2r_2)$ as a function of the parameters κt and γ .

$c_1r_1c_2$ can be written as

$$\rho_{c_1r_1c_2}(t) = \varphi_1(t) + \varphi_2(t), \quad (21)$$

where the non-normalized pure state components are $|\varphi_1(t)\rangle = \alpha\cos(\gamma/2)|000\rangle - \beta\sin(\gamma/2)\xi|001\rangle + \alpha\sin(\gamma/2)\chi|010\rangle + \alpha\sin(\gamma/2)\xi|100\rangle + \beta\cos(\gamma/2)\xi^2|101\rangle + \beta\cos\xi\chi|011\rangle$ and $|\varphi_2(t)\rangle = \beta\sin(\gamma/2)\chi|000\rangle - \beta\cos(\gamma/2)\chi^2|010\rangle - \beta\cos(\gamma/2)\xi\chi|100\rangle$, respectively. It is obvious that the $|\varphi_2\rangle$ is a separable state and its three-tangle is zero. Moreover, for the component $|\varphi_1\rangle$, we can derive $\tau(\varphi_1) = 0$. So, the decomposition in Eq. (21) is the optimal and the mixed state three-tangle $\tau_3(\rho_{c_1r_1c_2})$ is zero. Similarly, we can obtain that all the other mixed state three-tangles $\tau_3(\rho_{ijk})$ are zero.

Although all the $\tau_3(\rho_{ijk})$ are zero in the entanglement evolution, the three-qubit states are still entangled in the qubit-block form [30, 31], which is not equivalent to the mixed state three-tangle and can not be accounted for the two-qubit entanglement. The qubit-block entanglement characterizes the genuine three-qubit entanglement under bipartite cut between a qubit and a block of qubits, and can be defined as $E_{q-B}(\rho_{ijk}) = C_{i|jk}^2 - C_{ij}^2 - C_{ik}^2$, in which the $C_{i|jk}$ quantifies bipartite entanglement between the qubits i and jk . For the subsystems $c_1c_2r_2$ and $r_1c_2r_2$, their qubit-block entanglement are

$$\begin{aligned} E_{q-B}(\rho_{c_1|c_2r_2}) &= C_{c_1|c_2r_2}^2 - C_{c_1c_2}^2 - C_{c_1r_2}^2 \\ E_{q-B}(\rho_{r_1|c_2r_2}) &= C_{r_1|c_2r_2}^2 - C_{r_1r_2}^2 - C_{c_2r_1}^2, \end{aligned} \quad (22)$$

where $C_{c_1|c_2r_2}^2 = 4\alpha^2\beta^2\xi^2$, $C_{r_1|c_2r_2}^2 = 4\alpha^2\beta^2\chi^2$, and the expressions of two-qubit concurrences C_{ij}^2 are given in Eqs. (12), (14) and (15).

In Fig. 3, we plot the qubit-block entanglement as a function of the parameters κt and γ , where the initial state parameters are chosen as $\alpha = 1/\sqrt{10}$ and $\beta = 3/\sqrt{10}$. For a given value of the γ , the qubit-block entanglement $E_{c_1|c_2r_2}$ (in Fig. 3(a)) increases first with the time κt , and then decreases with the κt after attaining to its maximal value. Along with the increase of the γ , the maximal value of $E_{c_1|c_2r_2}$ decreases. For the qubit-block entanglement $E_{r_1|c_2r_2}$ (in Fig. 3(b)), the trend

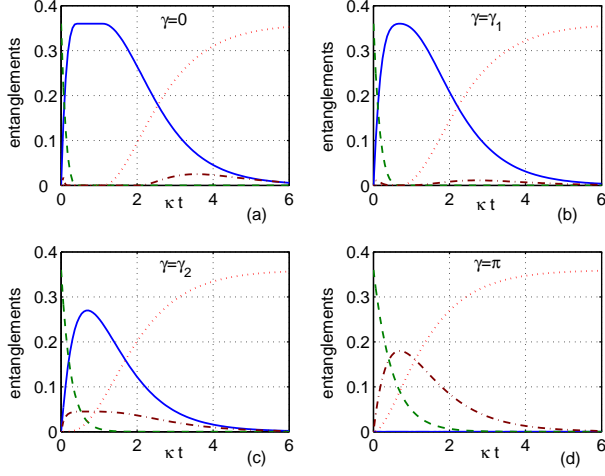


FIG. 4: (Color online) Entanglement evolutions modulated by different parameter values: (a) $\gamma = 0$, (b) $\gamma = \gamma_1$, (c) $\gamma = \gamma_2$ and (d) $\gamma = \pi$. The entanglements $C_{c_1c_2}^2$, $C_{r_1r_2}^2$, $C_{c_1r_2}^2 + C_{c_2r_1}^2$ and E_{BB} are represented by the green dashed line, red dotted line, brown dot-dashed line and blue solid line, respectively.

of entanglement evolution is similar. In Refs. [20, 31], it is pointed out that the qubit-block entanglement comes from the genuine multipartite entanglement in the enlarged pure state system. Here, for the multipartite cavity-reservoir system, we can derive the following relation

$$E_{BB}(\Psi_t) = E_{q-B}(\rho_{c_1|c_2r_2}(t)) + E_{q-B}(\rho_{r_1|c_2r_2}(t)), \quad (23)$$

which means that the qubit-block entanglement comes from the genuine block-block entanglement in the composite system.

V. ENTANGLEMENT TRANSFER AND ENTANGLEMENT TRANSITION UNDER THE LU OPERATION

Because the evolution $U_{c_1r_1}(\hat{H}, t) \otimes U_{c_2r_2}(\hat{H}, t)$ are two local unitary operations under the partition $c_1r_1|c_2r_2$, the bipartite entanglement $C_{c_1r_1|c_2r_2}^2$ is invariant, and the following relation holds

$$C_{c_1r_1|c_2r_2}^2(\Psi_t) = E_{BB}(t) + \sum C_{i'j'}^2(t) = 4\alpha^2\beta^2, \quad (24)$$

where $i' \in \{c_1, r_1\}$ and $j' \in \{c_2, r_2\}$, respectively. Therefore, in the multipartite cavity-reservoir system, we can characterize the entanglement evolution under a unified framework, where the two qubit entanglement transfer is quantified by the concurrence and the multipartite entanglement transition is quantified by the block-block entanglement E_{BB} . In Fig. 4, the entanglement evolution modulated by different value of γ is plotted, where $\gamma_1 = 2\arccos(\sqrt{2/3})$, $\gamma_2 = 2\arccos(\sqrt{1/3})$, and the initial state parameters are $\alpha = 1/\sqrt{10}$ and $\beta = 3/\sqrt{10}$.

In Fig. 4(a), the parameter is chosen as $\gamma = 0$, which corresponds to the symmetric initial state. In the time inter-

val $\kappa t \in (0, \ln(3/2))$, a part of the initial photon-photon entanglement $C_{c_1c_2}^2$ first transfers to the subsystems c_1r_2 and c_2r_1 (the brown dot-dashed line where a factor 5 is multiplied), then the remaining photon-photon entanglement and the cavity-reservoir entanglement transition completely to the genuine block-block entanglement E_{BB} (the blue solid line). Along with the time evolution, the block-block entanglement keeps invariant and is immune to the cavity-reservoir interaction in the time interval $\kappa t \in [\ln(3/2), \ln 3]$. Finally, when $\kappa t \in (\ln 3, 6]$, the multipartite entanglement E_{BB} transitions to the two-qubit reservoir-reservoir entanglement (the red dotted line) and the cavity-reservoir entanglement. When the parameter $\gamma \in (0, \gamma_1]$, the trends of bipartite and multipartite entanglement evolutions are similar to those when $\gamma = 0$, but the immune region of the block-block entanglement decreases with the parameter and the other evolution regions extend. In Fig. 4(b), the parameter is chosen as $\gamma = \gamma_1$, where the plateau region of the E_{BB} changes to a point ($\kappa t = \ln 2$) and the procedures of entanglement transfer and entanglement transition need more time.

When the parameter $\gamma \in (\gamma_1, \gamma_2]$, the initial photon entanglement transfers to not only the subsystems c_1r_2 and c_2r_1 but also the subsystem r_1r_2 , and the two-qubit entanglement can not transition completely to the genuine block-block entanglement. As shown in Fig. 4(c), the entanglement transfer and entanglement transition are plotted when $\gamma = \gamma_2$. Along with the increase of the γ , the decay of the photon entanglement slows down and the transfer ratio of the two-qubit entanglement increases. At the same time, the transition ratio of the block-block entanglement decreases. In Fig. 4(d), the parameter is chosen as $\gamma = \pi$, in which the transition between the two-qubit entanglement and the multipartite entanglement disappears, and the entanglement evolution consists of only two-qubit entanglement transfer.

It should be pointed out that, in the unified framework of entanglement evolution, the entanglement monotone property of E_{BB} is based on the numerical analysis on the average multipartite entanglement E_{ms} [25]. The analytic proof is still an open problem.

VI. DISCUSSION AND CONCLUSION

In the more general case, an arbitrary initial state has the form $|\Psi_0^g\rangle = (\alpha_1|00\rangle + \alpha_2|01\rangle + \alpha_3|10\rangle + \alpha_4|11\rangle)_{c_1c_2}|00\rangle_{r_1r_2}$, which corresponds to two LU operations $U_{c_1}(\zeta_1, \eta_1, \gamma_1, \delta_1) \otimes U_{c_2}(\zeta_2, \eta_2, \gamma_2, \delta_2)$ acting on the symmetric initial state. In this case, the analytical characterization for the entanglement evolution is not available so far. However, the correlation between the ESD of cavity photons and the ESB of reservoirs still holds. This is because we can deduce the relation

$$\rho_{c_1c_2}^g(\xi, \chi) = S_{\xi \leftrightarrow \chi}[\rho_{r_1r_2}^g(\xi, \chi)], \quad (25)$$

where the evolution $U_{cr}(\hat{H}, t)|10\rangle = \xi|10\rangle + \chi|01\rangle$ is used. Based on this relation, we can obtain that when the ESD of cavity photons occurs at the time $t_{ESD} = t_0$, the ESB of reservoirs will necessarily happen at the time $t_{ESB} =$

$-(1/\kappa)\ln[1 - \exp(-\kappa t_0)]$. Moreover, the entanglement evolution is restricted by the monogamy relation

$$C_{c_1 r_1 | c_2 r_2}^2(|\Psi_0^g\rangle) \geq C_{c_1 c_2}^2(t) + C_{r_1 r_2}^2(t) + C_{c_1 r_2}^2(t), \\ + C_{c_2 r_1}^2(t) \quad (26)$$

and the multipartite entanglement can be indicated by the two-qubit residual entanglement $M_{c_1 r_1}$ [20, 32].

The entanglement evolution with the asymmetric initial state is worth to consider for other physical systems, for example, the atoms systems [3], quantum dots and spin chains etc. [33–35]. Moreover, the dissipative entanglement evolution has close relation with the type of the noise environment. Therefore, the non-Markovian environment, the correlated noises, and some operator channels [36–39] are also worth to study in future.

In conclusion, we have investigated the entanglement evolution of multipartite cavity-reservoir systems with the asymmetric initial state. It is shown that there is only one parameter in the LU operation affecting the entanglement dynamics, which can delay the ESD of the photons, advance the ESB of the reservoirs, change the evolution route of bipartite entanglement, and suppress the multipartite entanglement. However, the correlation between the ESD and the ESB still holds. Furthermore, by defining the block-block entanglement, we analyze the multipartite entanglement evolution in the composite system, which allows us to study quantitatively both the entanglement transfer and the entanglement transition within a unified framework. Finally, the entanglement evolution with an arbitrary initial state is discussed.

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Appendix

We first prove Eq. (5). The output state under the time evolution is

$$|\Phi_t\rangle = U_{c_1 r_1}(\hat{H}, t) \otimes U_{c_2 r_2}(\hat{H}, t) |\Phi_0^a\rangle \\ = U_{c_1} U_{c_1}^\dagger U_{c_1 r_1}(\hat{H}, t) \otimes U_{c_2 r_2}(\hat{H}, t) [U_{c_1} |\Phi_0\rangle] \quad (27)$$

where, in the second equation, the identity operator $I = U_{c_1} U_{c_1}^\dagger$ is inserted. Substituting the U_{c_1} with the expression $R_z(\eta) R_y(\gamma) R_z(\delta)$ (we neglect the global phase $e^{i\zeta}$), we can obtain

$$|\Phi_t\rangle = U_l R_z^\dagger(\eta) U_{c_1 r_1}(\hat{H}, t) R_z(\eta)_{c_1} \otimes U_{c_2 r_2}(\hat{H}, t) \\ [R_y(\gamma)_{c_1} R_z(\delta)_{c_1} |\Phi_0\rangle] \quad (28) \\ \simeq U_{c_1 r_1}(\hat{H}', t) \otimes U_{c_2 r_2}(\hat{H}, t) [R_y(\gamma)_{c_1} R_z(\delta)_{c_1} |\Phi_0\rangle]$$

where the symbol \simeq means the quantum states on the two sides are equivalent up to the local unitary operation $U_l = U_{c_1} R_z^\dagger(\delta)_{c_1} R_y^\dagger(\gamma)_{c_1}$ (note that entanglement is invariant under local unitary transformation), and we use the relation $R_z^\dagger(\eta)_{c_1} U_{c_1 r_1}(\hat{H}, t) R_z(\eta)_{c_1} = U_{c_1 r_1}(\hat{H}', t)$ with $\hat{H}' = R_z^\dagger(\eta)_{c_1} \hat{H} R_z(\eta)_{c_1}$. Due to the symmetric property of initial state $|\Phi_0\rangle$, we have the relation $R_z(\delta)_{c_1} |\Phi_0\rangle = R_z(\delta)_{c_2} |\Phi_0\rangle$. Then the output state can be expressed further as

$$|\Phi_t\rangle = U_{c_1 r_1}(\hat{H}', t) \otimes R_z(\delta)_{c_2} R_z^\dagger(\delta)_{c_2} U_{c_2 r_2}(\hat{H}, t) R_z(\delta)_{c_2} \\ [R_y(\gamma)_{c_1} |\Phi_0\rangle] \\ \simeq U_{c_1 r_1}(\hat{H}', t) \otimes U_{c_2 r_2}(\hat{H}'', t) [R_y(\gamma)_{c_1} |\Phi_0\rangle], \quad (29)$$

where we insert the identity operator $R_z(\delta)_{c_2} R_z^\dagger(\delta)_{c_2} = I$ in the first equation and use the relation $\hat{H}'' = R_z^\dagger(\delta)_{c_2} \hat{H} R_z(\delta)_{c_2}$ in the second equation.

Next, we will prove the effects of \hat{H}' and \hat{H}'' are equivalent to that of \hat{H} in the entanglement evolution. Because the Hilbert space of subsystem $c_1 r_1$ is spanned by $|00\rangle, |01\rangle, |10\rangle$, the creation and annihilation operators are

$$\hat{a}^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } \hat{a} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad (30)$$

respectively. With these expressions, the rotation operator $R_z(\eta)$ can be rewritten as

$$R_z(\eta) = A \cdot \hat{a}^\dagger \hat{a} + B, \quad (31)$$

where the coefficients $A = \exp(i\eta/2) - \exp(-i\eta/2)$ and $B = \exp(-i\eta/2)$, respectively. After substituting the expression of $R_z(\eta)$ into the Hamiltonian \hat{H}' , we can derive

$$\hat{H}' = R_z^\dagger(\eta)_{c_1} \hat{H} R_z(\eta)_{c_1} \\ = (A \cdot \hat{a}^\dagger \hat{a} + B)^\dagger [\hbar \omega \hat{a}^\dagger \hat{a} + \hbar \sum_{k=1}^N \omega_k \hat{b}_k^\dagger \hat{b}_k \\ + \hbar \sum_{k=1}^N g_k (\hat{a} \hat{b}_k^\dagger + \hat{b}_k \hat{a}^\dagger)] (A \cdot \hat{a}^\dagger \hat{a} + B) \\ = \hbar \omega \hat{a}^\dagger \hat{a} + \hbar \sum_{k=1}^N \omega_k \hat{b}_k^\dagger \hat{b}_k + \hbar \sum_{k=1}^N g_k (\hat{a} \hat{b}_k^\dagger \cdot e^{i\eta} \\ + \hat{b}_k \hat{a}^\dagger \cdot e^{-i\eta}) \\ = V_{r_1}^\dagger(\eta) \hat{H} V_{r_1}(\eta), \quad (32)$$

where $V_r(\eta) = \text{diag}\{1, \exp(-i\eta)\}$, and we used the relations $\hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} = \hat{a}^\dagger \hat{a} = \hat{N}$ and $\hat{a}^\dagger \hat{N} \hat{b}_k = 0$ [40]. Similarly, for the Hamiltonian \hat{H}'' , we can obtain

$$\hat{H}'' = R_z^\dagger(\delta)_{c_2} \hat{H} R_z(\delta)_{c_2} = V_{r_2}^\dagger(\delta) \hat{H} V_{r_2}(\delta) \quad (33)$$

with $V_{r_2}(\delta) = \text{diag}\{1, \exp(-i\delta)\}$. Therefore, the output state in Eq. (29) can be written as

$$|\Phi_t\rangle = V_{r_1}^\dagger(\eta) V_{r_2}^\dagger(\delta) U_{c_1 r_1}(\hat{H}, t) \otimes U_{c_2 r_2}(\hat{H}, t) \\ [R_y(\gamma)_{c_1} V_{r_1}(\eta) V_{r_2}(\delta) |\Phi_0\rangle]. \quad (34)$$

In the above equation, the local unitary operation $V_{r_1}^\dagger(\eta)V_{r_2}^\dagger(\delta)$ does not change the entanglement evolution. Moreover, due to the reservoirs being in the vacuum state, we have $V_{r_1}(\eta)V_{r_2}(\delta)|\Phi_0\rangle = |\Phi_0\rangle$. Therefore, the effective output state in the entanglement evolution has the form

$$|\Psi_t\rangle = U_{c_1r_1}(\hat{H}, t) \otimes U_{c_2r_2}(\hat{H}, t)[R_y(\gamma)_{c_1}|\Phi_0\rangle]. \quad (35)$$

This means that, for the asymmetric initial state modulated by an arbitrary LU operation $U_{c_1}(\zeta, \eta, \gamma, \delta)$, the entanglement evolution is only sensitive to the rotation $R_y(\gamma)$.

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- [1] R. Horodecki, P. Horodecki, M. Horodecki and K. Horodecki, *Rev. Mod. Phys.* **81**, 865 (2009).
- [2] M. B. Plenio and S. Virmani, *Quantum Inf. Comput.* **7**, 1 (2007).
- [3] Ting Yu and J. H. Eberly, *Phys. Rev. Lett.* **93**, 140404 (2004); *ibid.* **97**, 140403 (2006).
- [4] K. Życzkowski, P. Horodecki, M. Horodecki, and R. Horodecki, *Phys. Rev. A* **65**, 012101 (2001).
- [5] A. K. Rajagopal and R. W. Rendell, *Phys. Rev. A* **63**, 022116 (2001).
- [6] S. Daffer, K. Wodkiewicz, and J. K. McIver, *Phys. Rev. A* **67**, 062312 (2003).
- [7] P. J. Dodd and J. J. Halliwell, *Phys. Rev. A* **69**, 052105 (2004).
- [8] M. Ban, *J. Phys. A*, **39**, 1927 (2006).
- [9] M. F. Santos, P. Milman, L. Davidovich, and N. Zagury, *Phys. Rev. A* **73**, 040305 (2006).
- [10] L. Derkacz and L. Jakóbczyk, *Phys. Rev. A* **74**, 032313 (2006).
- [11] Z. Sun, X. Wang, and C. P. Sun, *Phys. Rev. A* **75**, 062312 (2007).
- [12] Z.-G. Li, F.-S. Fei, Z. D. Wang and W.-M. Liu, *Phys. Rev. A* **79**, 024303 (2009).
- [13] Z. Liu and H. Fan, *Phys. Rev. A* **79**, 064305 (2009).
- [14] Q.-J. Tong, J.-H. An, H.-G. Luo, and C. H. Oh, *Phys. Rev. A* **81**, 052330 (2010).
- [15] Y. S. Weinstein, *Phys. Rev. A* **82**, 032326 (2010).
- [16] T. Yu and J. H. Eberly, *Science* **323**, 598 (2009).
- [17] M. P. Almeida, F. de Melo, M. Hor-Meyll, A. Salles, S. P. Walborn, P. H. Souto Ribeiro, L. Davidovich, *Science* **316**, 579 (2007).
- [18] J. Laurat, K. S. Choi, H. Deng, C. W. Chou, and H. J. Kimble, *Phys. Rev. Lett.* **99**, 180504 (2007).
- [19] C. E. López, G. Romero, F. Lastra, E. Solano, and J. C. Retamal, *Phys. Rev. Lett.* **101**, 080503 (2008).
- [20] Y.-K. Bai, M.-Y. Ye, and Z. D. Wang, *Phys. Rev. A* **80**, 044301 (2009).
- [21] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000), p20.
- [22] W. K. Wootters, *Phys. Rev. Lett.* **80**, 2245 (1998).
- [23] The square root of the eigenvalues of matrix $R_{c_1c_2}$ are $\sqrt{\lambda_1} = \alpha\beta\xi^2 + \sqrt{\beta^2\xi^4(\alpha^2 + \beta^2cr^4\chi^4)}$, $\sqrt{\lambda_2} = -\alpha\beta\xi^2 + \sqrt{\beta^2\xi^4(\alpha^2 + \beta^2cr^4\chi^4)}$, and $\sqrt{\lambda_3} = \sqrt{\lambda_4} = \beta^2\xi^2\chi^2cr^2$ with $cr = \cos(\gamma/2)$. The square root of the eigenvalues of matrix $R_{r_1r_2}$ are same to those of matrix $R_{c_1c_2}$ after exchanging the parameters ξ and χ .
- [24] A. R. R. Carvalho, F. Mintert, and A. Buchleitner, *Phys. Rev. Lett.* **93**, 230501 (2004).
- [25] Y.-K. Bai, D. Yang, and Z. D. Wang, *Phys. Rev. A* **76**, 022336 (2007).
- [26] Y.-K. Bai and Z. D. Wang, *Phys. Rev. A* **77**, 032313 (2008).
- [27] X.-J. Ren, W. Jiang, X. Zhou, Z. W. Zhou, and G. C. Guo, *Phys. Rev. A* **78**, 012343 (2008).
- [28] A. Wong and N. Christensen, *Phys. Rev. A* **63**, 044301 (2001).
- [29] V. Coffman, J. Kundu, and W. K. Wootters, *Phys. Rev. A* **61**, 052306 (2000).
- [30] R. Lohmayer, A. Osterloh, J. Siewert, and A. Uhlmann, *Phys. Rev. Lett.* **97**, 260502 (2006).
- [31] Y.-K. Bai, M.-Y. Ye, and Z. D. Wang, *Phys. Rev. A* **78**, 062325 (2008).
- [32] T. J. Osborne and F. Verstraete, *Phys. Rev. Lett.* **96**, 220503 (2006).
- [33] D. Loss and D. V. DiVincenzo, *Phys. Rev. A* **57**, 120 (1998).
- [34] S.-S. Li, G.-L. Long, F.-S. Bai, S.-L. Feng, and H.-Z. Zheng, *Proc. Natl. Acad. Sci. U.S.A.* **98**, 11847 (2001).
- [35] X. Wang and Z. D. Wang, *Phys. Rev. A* **73**, 064302 (2006).
- [36] B. Bellomo, R. Lo Franco, and G. Compagno, *Phys. Rev. Lett.* **99**, 160502 (2007).
- [37] E. Novais, E. R. Mucciolo, and H. U. Baranger, *Phys. Rev. A* **78**, 012314 (2008).
- [38] T. Konrad, F. De Melo, M. Tiersch, C. Kasztelan, A. Aragao, and A. Buchleitner, *Nature Physics* **4**, 99 (2008).
- [39] X.-B. Wang, Z.-W. Yu and J.-Z. Hu, arXiv:1001.0156.
- [40] Due to the space of system c_1r_1 is spanned by $\{|00\rangle, |01\rangle, |10\rangle\}$, we have $(\hat{a}^\dagger\hat{a} - 1)\hat{a}^\dagger\hat{a} = 0 \Rightarrow \hat{N}^2 = \hat{N}$. Similarly, according to the space of system c_1r_1 , we have $\hat{a}^\dagger\hat{b}\hat{b}^\dagger\hat{b}\hat{N} = 0$. Combining it with the commutation relation of \hat{b} and \hat{b}^\dagger , we can derive $\hat{a}^\dagger\hat{N}\hat{b}_k = 0$.